ΣΕΠΤΕΜΒΡΙΟΣ 2015

ΘΕΜΑ 1ο

i) $f(x) = x^4 - x^2 + x$, στο $[-1, 1]$

Η βοπ της $f(x)$ στον P_3 :

$$\varphi_3^*(x) = f(x) - \frac{1}{2^3} T_4(x) \quad (1)$$

$$T_4(x) = 2x\, T_3(x) - T_2(x) \quad (2)$$

$$T_3(x) = 2x\, T_2(x) - T_1(x) \quad (3)$$

$$T_2(x) = 2x\, T_1(x) - T_0(x) = 2x^2 - 1$$

$$(3) \longrightarrow \quad T_3(x) = 2x(2x^2-1) - x = 4x^3 - 3x$$

$$(2) \longrightarrow \quad T_4(x) = 2x(4x^3-3x) - (2x^2-1) = 8x^4 - 8x^2 + 1$$

$$(1) \longrightarrow \quad p_3^*(x) = x^4 - x^2 + x - \frac{1}{8}(8x^4 - 8x^2 + 1) \Rightarrow$$

$$\Rightarrow \boxed{p_3^*(x) = x - \frac{1}{8}}$$

ii) Αφού $p_3^*(x)$ είναι 1ου βαθμού τότε

$$p_3^*(x) = p_2^*(x) = p_1^*(x).$$

ΘΕΜΑ 2°:

x_i	-2	-1	0	1	2
f_i	-2	-3	-2	-1	2

, $X_\sigma = \{-2, -1, 0\}$.

Θα βρούμε την βοπ $p_\sigma(x) \in P_1$.

άρα $p_\sigma(x) = \alpha_0 + \alpha_1 x$:

$$\begin{bmatrix} -1 & 1 & -2 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix} \xrightarrow{\text{Gauss}} \begin{bmatrix} -1 & 1 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \lambda \\ \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 0 \end{bmatrix}$$

$\rightarrow \alpha_1 = 0$, $\alpha_0 = -\frac{5}{2}$, $-\lambda - \frac{5}{2} = -2 \Rightarrow \lambda = -\frac{1}{2}$

Άρα, $p_\sigma(x) = -\frac{5}{2}$. Εξετάζουμε αν το X_σ είναι πράγματι ένα σύνολο εναλλασσόμενων σημείων $e_\sigma(x) = f(x) - p_\sigma(x)$, για τα δοθέντα $x = x_i$, $\forall i = 0,1,2,3,4$.

- $e_\sigma(-2) = f(-2) - p_\sigma(-2) = -2 + \frac{5}{2} = \frac{1}{2}$
- $e_\sigma(-1) = f(-1) - p_\sigma(-1) = -3 + \frac{5}{2} = -\frac{1}{2}$
- $e_\sigma(0) = f(0) - p_\sigma(0) = -2 + \frac{5}{2} = \frac{1}{2}$ $\Leftarrow$ ΕΞΟΔΟΣ
- $e_\sigma(1) = f(1) - p_\sigma(1) = -1 + \frac{5}{2} = \frac{3}{2}$
- $e_\sigma(2) = f(2) - p_\sigma(2) = 2 + \frac{5}{2} = \frac{9}{2}$ $\Leftarrow$ ΕΙΣΟΔΟΣ

$M = \frac{9}{2} > \rho_\sigma = |\lambda| = \frac{1}{2}$ Άρα $X_\mu = \{-2, -1, 2\}$

Η υποψήφια ΒΟΠ τώρα είναι η εξής:

$P_\mu(x) = \alpha_0' + \alpha_1' x$

$$\begin{bmatrix} -1 & 1 & -2 \\ 1 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \lambda' \\ \alpha_0' \\ \alpha_1' \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} \xrightarrow{Gauss} \begin{bmatrix} -1 & 1 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \lambda' \\ \alpha_0' \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 4 \end{bmatrix}$$

$\alpha_1' = 1$, $\alpha_0' = \dfrac{3-5}{2} = -1$, $-\lambda - 1 - \cancel{2} = \cancel{-2} \Rightarrow \lambda = -1$

Άρα, $P_M(x) = -1 + x$

- $e_\mu(-2) = -2 - (-1-2) = 1$ ✓
- $e_M(-1) = -3 - (-1-1) = -1$ ✓
- $e_M(0) = -2 - (-1+0) = -1$
- $e_M(1) = -1 - (-1+1) = -1$
- $e_M(2) = 2 - (-1+2) = 1$ ✓

$\Bigg\}$ Εναλλασσόμενο σύνολο σημείων το $X_M = \{-2, -1, 2\}$ αφού $M' = 1 = \rho_M$.

ΘΕΜΑ 3°:

x_i	-2	-1	1	2
f_i	2	1	1	2

$\alpha_k = \dfrac{(xP_k, P_k)}{(P_k, P_k)}$, $\beta_k = \dfrac{(P_k, P_k)}{(P_{k-1}, P_{k-1})}$

$q_2^*(x) = \lambda_0 P_0(x) + \lambda_1 P_1(x) + \lambda_2 P_2(x) \in P_2$

$P_{k+1}(x) = (x - \alpha_k) P_k(x) - \beta_k P_{k-1}$ (*)

$P_0(x) = 1$, $P_1(x) = x - \dfrac{(x, P_0)}{(P_0, P_0)} P_0(x) =$

$$= x - \frac{\sum_{i=1}^{4} x_i \cancel{P_0(x_i)}^{1}}{\sum_{i=1}^{4} \cancel{(P_0(x_i))^2}^{1}} \cdot 1 = x - \frac{0}{4} = x .$$

Στην (*) τοτε εχουμε:

$$P_2(x) = (x - \alpha_1) P_1(x) - \beta_1 P_0(x) \quad , \quad (**)$$

$$\alpha_1 = \frac{(x P_1, P_1)}{(P_1, P_1)} = \frac{\sum_{i=1}^{4} x_i (P_1)^2}{\sum_{i=1}^{4} (P_1)^2} = \frac{\sum_{i=1}^{4} (x_i)^3}{\sum_{i=1}^{4} (x_i)^2} = 0$$

$$\beta_1 = \frac{(P_1, P_1)}{(P_0, P_0)} = \frac{\sum_{i=1}^{4} (P_1)^2}{\sum_{i=1}^{4} (P_0)^2} = \frac{\sum_{i=1}^{4} (x_i)^2}{4} = \frac{10}{4} = \frac{5}{2}$$

Αρα, η (**) γίνεται

$$P_2(x) = x \cdot x - \frac{5}{2} \cdot 1 = x^2 - \frac{5}{2} .$$

Τελος,

$$\lambda_0 = \frac{(f, P_0)}{(P_0, P_0)} = \frac{\sum_{i=1}^{4} f(x_i)}{4} = \frac{2+1+1+2}{4} = \frac{3}{2}$$

$$\lambda_1 = \frac{(f, P_1)}{(P_1, P_1)} = \frac{\sum_{i=1}^{4} f(x_i) P_1}{\sum_{i=1}^{4} (P_1)^2} = \frac{\sum_{i=1}^{4} f(x_i) x_i}{\sum_{i=1}^{4} (x_i)^2} = \frac{2 \cdot (-2) + 1(-1) + 1 \cdot 1 + 2 \cdot 2}{10} = 0$$

$$\lambda_2 = \frac{(f, P_2)}{(P_2, P_2)} = \frac{\sum_{i=1}^{4} f(x_i) P_2}{\sum_{i=1}^{4} (P_2)^2} = \frac{\sum_{i=1}^{4} f(x_i)(x_i^2 - \frac{5}{2})}{\sum_{i=1}^{4} (x_i - \frac{5}{2})^2} =$$

$$= \frac{2 \cancel{\left(4 - \frac{5}{2}\right)}^{3/2} + 1 \cancel{\left(1 - \frac{5}{2}\right)}^{-3/2} + 1 \cdot \cancel{\left(1 - \frac{5}{2}\right)}^{-3/2} + 2 \cancel{\left(4 - \frac{5}{2}\right)}^{3/2}}{\underbrace{\cancel{\left(4 - \frac{5}{2}\right)^2}}_{9/2} + \underbrace{\cancel{\left(1 - \frac{5}{2}\right)^2}}_{9/2} + \underbrace{\cancel{\left(1 - \frac{5}{2}\right)^2}}_{9/2} + \underbrace{\cancel{\left(4 - \frac{5}{2}\right)^2}}_{9/2}} = \frac{3}{18}$$

Αρα, $q_2^*(x) = \dfrac{3}{2} + \dfrac{3}{18}\left(x^2 - \dfrac{5}{2}\right) = \dfrac{3}{18} x^2 + \dfrac{3}{2} - \dfrac{15}{36} = \dfrac{3}{18} x^2 + \dfrac{19}{36}$.

ΘΕΜΑ 4ο

x_i	-2	-1	1	2
f_i	1	0	0	1

$f'(-2) = -\frac{4}{3}$, $f'(2) = \frac{4}{3}$.

$$\Delta x_0 = x_1 - x_0 = -1+2 = 1 \qquad \Delta f_0 = f_1 - f_0 = 0-1 = -1$$

$$\Delta x_1 = x_2 - x_1 = 1+1 = 2 \qquad \Delta f_1 = f_2 - f_1 = 0$$

$$\Delta x_2 = x_3 - x_2 = 2-1 = 1 \qquad \Delta f_2 = f_3 - f_2 = 1-0 = 1$$

$$\begin{bmatrix} 2(\Delta x_0 + \Delta x_1) & \Delta x_0 \\ \Delta x_2 & 2(\Delta x_1 + \Delta x_2) \end{bmatrix} \begin{bmatrix} s_1' \\ s_2' \end{bmatrix} = \begin{bmatrix} 3\left(\frac{\Delta x_1}{\Delta x_0}\Delta f_0 + \frac{\Delta x_0}{\Delta x_1}\Delta f_1\right) - \Delta x_1 \cdot f'(-2) \\ 3\left(\frac{\Delta x_2}{\Delta x_1}\Delta f_1 + \frac{\Delta x_1}{\Delta x_2}\Delta f_2\right) - \Delta x_1 \cdot f'(2) \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} s_1' \\ s_2' \end{bmatrix} = \begin{bmatrix} 3\left(\underbrace{2\cdot(-1) + \frac{1}{2}\cdot 0}_{-2}\right) + 2\frac{4}{3} \\ 3\left(\underbrace{\frac{1}{2}\cdot 0 + 2\cdot 1}_{2}\right) - 2\frac{4}{3} \end{bmatrix} = \begin{bmatrix} -6 + \frac{8}{3} \\ 6 - \frac{8}{3} \end{bmatrix} = \begin{bmatrix} -\frac{10}{3} \\ \frac{10}{3} \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \quad s' = \left[-\frac{2}{3}, \frac{2}{3}\right]^T.$$

Τύπος Hermite:

1) $x \in [-2,-1]$:

$$s_1(x) = \left((x+1)^2 + 2(x+2)(x+1)^2\right)\cdot 1 + \left((x+2)(x+1)^2\right)\cdot\left(-\frac{4}{3}\right) +$$

$$+ \left((x+2)^2 - 2(x+1)(x+2)^2\right)\cdot 0 + \left((x+1)(x+2)^2\right)\left(-\frac{3}{2}\right) = \ldots$$

2) $x \in [-1,1]$

$$s_2(x) = \left(\frac{(x-1)^2}{4} + 2\frac{(x+1)\cdot(x-1)^2}{8}\right)\cdot 0 + \left(\frac{(x+1)(x-1)^2}{4}\right)\cdot\left(-\frac{3}{2}\right) +$$

$$+ \left(\frac{(x+1)^2}{4} - 2\frac{(x-1)(x+1)^2}{8}\right)\cdot 0 + \left(\frac{(x-1)(x+1)^2}{4}\right)\cdot\frac{3}{2} = \ldots$$

3) $x \in [1,2]$:

$$S_3(x) = \left((x-2)^2 + 2(x-1)(x-2)^2\right)\cdot 0 + \left((x-1)(x-2)^2\right)\cdot \frac{3}{2} +$$
$$+ \left((x-1)^2 - 2(x-2)(x-1)^2\right)\cdot 1 + \left((x-2)\cdot(x-1)^2\right)\cdot \frac{4}{3} = \dots$$

Άρα, η συνάρτηση Spline είναι:

$$S(x) = \begin{cases} S_1(x), & x \in [-2,-1] \\ S_2(x), & x \in [-1,1] \\ S_3(x), & x \in [1,2]. \end{cases}$$